

Universally Utility-Maximizing Privacy Mechanisms

Arpita Ghosh*

Tim Roughgarden†

Mukund Sundararajan‡

March 20, 2009

Abstract

A mechanism for releasing information about a statistical database with sensitive data must resolve a trade-off between utility and privacy. Publishing fully accurate information maximizes utility while minimizing privacy, while publishing random noise accomplishes the opposite. Privacy can be rigorously quantified using the framework of *differential privacy*, which requires that a mechanism’s output distribution is nearly the same whether or not a given database row is included or excluded. The goal of this paper is strong and general utility guarantees, subject to differential privacy.

We pursue mechanisms that guarantee near-optimal utility to every potential user, independent of its side information (modeled as a prior distribution over query results) and preferences (modeled via a loss function). Our main result is: for each fixed count query and differential privacy level, there is a *geometric mechanism* M^* — a discrete variant of the simple and well-studied Laplace mechanism — that is *simultaneously expected loss-minimizing* for every possible user, subject to the differential privacy constraint. This is an extremely strong utility guarantee: *every* potential user u , no matter what its side information and preferences, derives as much utility from M^* as from interacting with a differentially private mechanism M_u that is optimally tailored to u . More precisely, for every user u there is an optimal mechanism M_u for it that factors into a user-independent part (the geometric mechanism M^*) followed by user-specific post-processing that can be delegated to the user itself.

The first part of our proof of this result characterizes the optimal differentially private mechanism for a fixed but arbitrary user in terms of a certain basic feasible solution to a linear program with constraints that encode differential privacy. The second part shows that all of the relevant vertices of this polytope (ranging over all possible users) are derivable from the geometric mechanism via suitable remappings of its range.

1 Introduction

Organizations including the census bureau, medical establishments, and Internet companies collect and publish statistical information [6, 19]. The census bureau may, for instance, publish the result of a query such as: “How many individuals have incomes that exceed \$100,000?”. An implicit

*Yahoo! Research, 2821 Mission College Boulevard, Santa Clara, CA. Email: arpita@yahoo-inc.com.

†Department of Computer Science, Stanford University, 462 Gates Building, 353 Serra Mall, Stanford, CA 94305. Supported in part by NSF CAREER Award CCF-0448664, an ONR Young Investigator Award, an AFOSR MURI grant, and an Alfred P. Sloan Fellowship. Email: tim@cs.stanford.edu.

‡Department of Computer Science, Stanford University, 470 Gates Building, 353 Serra Mall, Stanford, CA 94305. Supported by NSF Award CCF-0448664, a Stanford Graduate Fellowship and an internship at Yahoo! Research. Email: mukunds@cs.stanford.edu.

hope in this approach is that aggregate information is sufficiently anonymous so as not to breach the privacy of any individual. Unfortunately, publication schemes initially thought to be “private” have succumbed to privacy attacks [19, 17, 1], highlighting the urgent need for mechanisms that are *provably* private. The differential privacy literature [10, 8, 16, 18, 5, 7, 12, 4] has proposed a rigorous and quantifiable definition of privacy, as well as provably privacy-preserving mechanisms for diverse applications including statistical queries, machine learning, and pricing. Informally, for $\alpha \in [0, 1]$, a randomized mechanism is α -differentially private if changing a row of the underlying database—the data of a single individual—changes the probability of every mechanism output by at most an α factor. Larger values of α correspond to greater levels of privacy. Differential privacy is typically achieved by adding noise that scales with α . While it is trivially possible to achieve any level of differential privacy, for instance by always returning random noise, this completely defeats the original purpose of providing useful information. On the other hand, returning fully accurate results can lead to privacy disclosures [8]. *The goal of this paper is to identify, for each $\alpha \in [0, 1]$, the optimal (i.e., utility-maximizing) α -differentially private mechanism.*

2 Model

2.1 Differential Privacy

We consider databases with n rows drawn from a finite domain D . Every row corresponds to an individual. Two databases are *neighbors* if they coincide in $n - 1$ rows. A *count query* f takes a database $d \in D^n$ as input and returns the result $f(d) \in N = \{0, \dots, n\}$ that is the number of rows that satisfy a fixed, non-trivial predicate on the domain D . Such queries are also called predicate or subset-sum queries; they have been extensively studied in their own right [7, 4, 12, 5], and form a basic primitive from which more complex queries can be constructed [4].

A randomized mechanism with a (countable) range R is a function x from D^n to R , where x_{dr} is the probability of outputting the response r when the underlying database is d . For $\alpha \in [0, 1]$, a mechanism x is α -*differentially private* if the ratio x_{d_1r}/x_{d_2r} lies in the interval $[\alpha, 1/\alpha]$ for every possible output $r \in R$ and pair d_1, d_2 of neighboring databases¹. (We interpret $0/0$ as 1.) Intuitively, the probability of every response of the privacy mechanism — and hence the probability of a successful privacy attack following an interaction with the mechanism — is, up to a controllable α factor, independent of whether a given user “opts in” or “opts out” of the database.

A mechanism is *oblivious* if, for all $r \in R$, $x_{d_1r} = x_{d_2r}$ whenever $f(d_1) = f(d_2)$ — if the output distribution depends only on the query result. Most of this paper considers only oblivious mechanisms; for optimal privacy mechanism design, this is without loss of generality in a precise sense (see Section 6.2). The notation and definitions above simplify for oblivious mechanisms and count queries. We can specify an oblivious mechanism via the probabilities x_{ir} of outputting a response $r \in R$ for each query result $i \in N$; α -differential privacy is then equivalent to the constraint that the ratios $x_{ir}/x_{(i+1)r}$ lie in the interval $[\alpha, 1/\alpha]$ for every possible output $r \in R$ and query result $i \in N \setminus \{n\}$.

Example 2.1 (Geometric Mechanism) The α -*geometric mechanism* is defined as follows. When the true query result is $f(d)$, the mechanism outputs $f(d) + Z$. Z is a random variable distributed

¹ The usual definition of differential privacy requires this bound on the ratio for all subsets of the range. With a countable range, the two definitions are equivalent.

as a two-sided geometric distribution: $Pr[Z = z] = \frac{1-\alpha}{1+\alpha}\alpha^{|z|}$ for every integer z . This (oblivious) mechanism is α -differentially private because the probabilities of adjacent points in its range differ by an α factor and because the true answer to a count query differs by at most one on neighboring databases.

2.2 Utility Model

This paper pursues strong and general utility guarantees. Just as differential privacy guarantees protection against every potential attacker, independent of its side information, we seek mechanisms that guarantee near-optimal utility to *every* potential user, independent of its side information and preferences.

We now formally define preferences and side information. We model the preferences of a user via a *loss function* l ; $l(i, r)$ denotes the user’s loss when the query result is i and the mechanism’s (perturbed) output is r . We allow l to be arbitrary, subject only to being nonnegative, and nondecreasing in $|i - r|$ for each fixed i . For example, the loss function $|i - r|$ measures mean error, the implicit measure of (dis)utility in most previous literature on differential privacy. Two among the many other natural possibilities are $(i - r)^2$, which essentially measures variance of the error; and the binary loss function $l_{bin}(i, r)$, defined as 0 if $i = r$ and 1 otherwise.

We model the side information of a user as a prior probability distribution $\{p_i\}$ over the query results $i \in N$. This prior represents the beliefs of the user, which might stem from other information sources, previous interactions with the mechanism, introspection, or common sense. We emphasize that we are *not* introducing priors to weaken the definition of differential privacy; we use the standard definition of differential privacy (which makes no assumptions about the side information of an attacker) and use a prior only to discuss the *utility* of a (differentially private) mechanism to a potential user.

Now consider a user with a prior $\{p_i\}$ and loss function l and an oblivious mechanism x with range R . For a given input d with query result $i = f(d)$, the user’s expected loss is $\sum_{r \in R} x_{ir} \cdot l(i, r)$, where the expectation is over the coin flips internal to the mechanism. The user’s prior then yields a measure of the mechanism’s overall (dis)utility to the the user:

$$\sum_{i \in N} p_i \sum_{r \in R} x_{ir} \cdot l(i, r). \quad (1)$$

This is simply the expected loss over the coin tosses of the mechanism and the prior.² We can then define the *optimal* α -differentially private mechanism for a user as one that minimizes the user-specific objective function (1).

2.3 User Post-Processing

Could a single mechanism be good simultaneously for all users? A crucial observation for an affirmative answer is that a user has the power to post-process the output of a privacy mechanism, and that such post-processing can decrease the user’s expected loss.

²The central theorem of choice theory (e.g. [15, Chapter 6]) states that every preference relation over mechanisms that satisfies reasonable axioms (encoding “rationality”) can be modeled via *expected* utility, just as we propose. In particular, this theorem justifies the use of priors for expressing a rational user’s trade-off over possible query results.

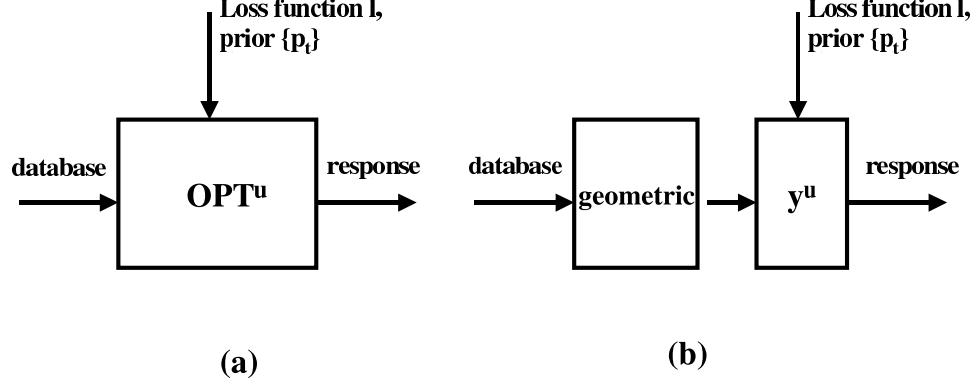


Figure 1: Theorem 3.1. For every rational user u , (a) can be factored into a user-independent part (the α -geometric mechanism) followed by a user-dependent post-processing step (the optimal remap y^u).

Example 2.2 (Post-Processing Decreases Loss) Fix a database size n that is odd. Consider a user with the binary loss function l_{bin} and prior $p_0 = p_n = 1/2$, $p_j = 0$ for all $j \in N \setminus \{0, n\}$, that interacts with the α -geometric mechanism. Without post-processing, i.e., when the user accepts the mechanism's outputs at face value, the user's expected loss is $(2 \cdot \alpha)/(1 + \alpha)$. If the user maps outputs of the geometric mechanism that are at least $(n + 1)/2$ to n and all other outputs to 0, it effectively induces a new mechanism with the much smaller expected loss of $\alpha^{(n+1)/2}/(1 + \alpha)$.

In general, a (randomized) *remap* of a mechanism x with range R is a probabilistic function y , with $y_{r'r}$ denoting the probability that the user reinterprets the mechanism's response $r' \in R$ as the response $r \in R$. A mechanism x and a remap y together induce a new (α -differentially private) mechanism $y \circ x$ with $(y \circ x)_{ir} = \sum_{r' \in R} x_{ir'} \cdot y_{r'r}$.

We assume that a (rational) user with prior p and loss function l , interacting with a publicly known mechanism x , employs a remap y that induces the mechanism $y \circ x$ that minimizes its expected loss (1) over all such remaps. It is well known (e.g. [15, Chapter 9]) and easy to see that, among all possible (randomized) remappings, the optimal one follows from applying Bayes rule and then minimizing expected loss. Precisely, for each response r of x , compute the posterior probability over query results: For every $i \in R$, $p(i|r) = (x_{ir} \cdot p_i)/(\sum_{i' \in R} x_{i'r} \cdot p_{i'})$. Then, choose the query result $i^* \in R$ that minimizes expected loss subject to the posterior and set $y_{ri^*} = 1$ and $y_{ri} = 0$ for $i \neq i^*$. This remap is simple, deterministic, and can be computed efficiently.

3 Main Result and Discussion

Our main result is that for every $\alpha \in [0, 1]$, the α -geometric mechanism is simultaneously optimal for every rational user.

Theorem 3.1 (Main Result) Let x^G denote the α -geometric mechanism for some database size $n \geq 1$ and privacy level $\alpha \in [0, 1]$, and let y^u denote an optimal remap of x^G for the user u with prior p and (monotone) loss function l . Then $y^u \circ x^G$ minimizes u 's expected loss (1) over all oblivious, α -differentially private mechanisms with range N .

This is an extremely strong utility-maximization guarantee: *every* potential user u , no matter what its side information and preferences, derives as much utility from the geometric mechanism as it does from interacting with a differentially private mechanism M_u that is optimally tailored to u . We reiterate that the prior from the utility model plays no role in the definition of privacy, which is the standard, worst-case (over adversaries with arbitrary side-information and intent) guarantee provided by differential privacy. We emphasize that while the geometric mechanism is user-independent (all users see the same distribution over responses), different users remap its responses in different ways, as informed by their individual prior distributions and loss functions. Rephrasing Theorem 3.1, for every user there is an optimal mechanism for it that factors into a user-independent part—the α -geometric mechanism—and a user-specific computation that can be delegated to the user. (See Figure 3.)

Theorem 3.1 shows how to achieve the same utility as a user-specific optimal mechanism without directly implementing one. Direct user-specific optimization would clearly involve several challenges. First, it would require advance knowledge or elicitation of user preferences, which we expect is impractical in most applications. And even if a mechanism was privy to the various preferences of its users, it would effectively need to answer the same query in different ways for different users, which in turn degrades its differential privacy guarantee.

In Theorem 3.1, the restriction to oblivious mechanisms is, in a precise sense, without loss of generality. (See Section 6.2.) The restriction to the range N effectively requires that the mechanism output is a legitimate query result for some database; this type of property is called “consistency” in the literature (e.g. [2]).

4 Related Work

Differential privacy is motivated in part by the provable impossibility of absolute privacy against attackers with arbitrary side information [8]. One interpretation of differential privacy is: no matter what prior distribution over databases a potential attacker has, its posterior after interacting with a differentially private mechanism is almost independent of whether a given user “opted in” or “opted out” of the database [10, 14]. Below we discuss the papers in the differential privacy literature closest to the present work; see [9] for a recent, thorough survey of the state of the field.

Dinur and Nissim [7] showed that for a database with n rows, answering $O(n \log^2 n)$ randomly chosen subset count queries with $o(\sqrt{n})$ error allows an adversary to reconstruct most of the rows of the database (a blatant privacy breach); see Dwork et al. [11] for a more robust impossibility result of the same type. Most of the differential privacy literature circumvents these impossibility results by focusing on interactive models where a mechanism supplies answers to only a sub-linear (in n) number of queries. Count queries (e.g. [7, 12]) and more general queries (e.g. [10, 18]) have been studied from this perspective.

Blum et al. [5] take a different approach by restricting attention to count queries that lie in a restricted class; they obtain non-interactive mechanisms that provide simultaneous good accuracy (in terms of worst-case error) for all count queries from a class with polynomial VC dimension. Kasiviswanathan et al. [13] give further results for privately learning hypotheses from a given class.

The use of abstract “utility functions” in McSherry and Talwar [16] has a similar flavor to our use of loss functions, though the motivations and goals of their work and ours are unrelated. Motivated by pricing problems, McSherry and Talwar [16] design differentially private mechanisms for queries that can have very different values on neighboring databases (unlike count queries);

they do not consider users with side information (i.e., priors) and do not formulate a notion of mechanism optimality (simultaneous or otherwise).

Finally, in recent and independent work, McSherry and Talwar (personal communication, October 2008) also apply linear programming theory in the analysis of privacy mechanisms. Again, their goal is different: they do not consider a general utility model, but instead ask how expected error must scale with the number of queries answered by a differentially private mechanism.

5 Proof of Main Result

This section proves Theorem 3.1. The proof has three high-level steps.

1. For a given user u , we formulate the problem of determining the differentially private mechanism that minimizes expected loss as a solution to a linear program (LP). The objective function of this LP is user-specific, but the feasible region is not.
2. We identify several necessary conditions met by every privacy mechanism that is optimal for some user.
3. For every privacy mechanism that satisfies these conditions, we construct a remap y such that $y \circ x^G = x$. By assumption, a rational user employs an “optimal remap” of x^G , so the mechanism induced by this map must be optimal for the user u .

Fix a database size n and a privacy level α . Theorem 3.1 is trivially true for the degenerate cases of $\alpha = 0, 1$. So, we assume that $\alpha \in (0, 1)$. For every fixed user with loss function l and prior p , the formulation of privacy constraints in Section 2.1 together with the objective function (1) yields the following LP whose solution is an optimal mechanism for this user.

User-specific LP:

$$\text{minimize} \quad \sum_{i \in N} p_i \sum_{r \in N} x_{ir} \cdot l(i, r) \quad (2)$$

$$x_{ir} - \alpha \cdot x_{(i+1)r} \geq 0 \quad \forall r \in N \setminus \{n\}, \forall i \in N \quad (3)$$

$$\alpha \cdot x_{ir} - x_{(i+1)r} \leq 0 \quad \forall r \in N \setminus \{n\}, \forall i \in N \quad (4)$$

$$\sum_{r \in R} x_{ir} = 1 \quad \forall i \in N \quad (5)$$

$$x_{ir} \geq 0 \quad \forall i \in N, \forall r \in N \quad (6)$$

Since the LP is bounded and feasible, we have the following (e.g. [3]).

Lemma 5.1 *Every user-specific LP has an optimal solution that is a vertex.*

For the rest of this section, fix a user with prior $\{p_i\}$ and a loss function $l(i, r)$ that is monotone in $|i - r|$ for every $i \in N$. Fix a mechanism x that is optimal for this user, and also a vertex of the polytope of the user-specific LP. Vertices can be uniquely identified by the set of all constraints that are tight at the vertex. This motivates us to characterize the *state of constraints* (slack or tight) of mechanisms that are optimal for some user.

We now begin the second step of the proof. We will view x as a $(n+1) \times (n+1)$ -matrix where rows correspond to query results (inputs) and columns correspond to query responses (outputs).

input/output	0	1	2	3	4	5
0	2/3	0	1/4	1/24	1/48	1/48
1	1/3	0	1/2	1/12	1/24	1/24
2	1/6	0	1/2	1/6	1/12	1/12
3	1/12	0	1/4	1/3	1/6	1/6
4	1/24	0	1/8	1/6	1/3	1/3
5	1/48	0	1/16	1/12	1/6	2/3

Figure 2: The database size n is 5. Figure shows an optimal $1/2$ -differentially private mechanism for a user with prior $1/4, 0, 1/4, 0, 1/4, 1/4$ on the six possible results and the loss function $l(i, r) = |i - r|^{1.5}$

input/output	0	1	2	3	4	5
0	↓	Z	↑	↑	↑	↑
1	↓	Z	S	↑	↑	↑
2	↓	Z	↓	↑	↑	↑
3	↓	Z	↓	↓	↑	↑
4	↓	Z	↓	↓	↓	↑

Figure 3: The constraint matrix for the mechanism from Figure 2.

We state our necessary conditions in terms of an $n \times (n+1)$ *constraint* matrix C associated with the mechanism x . Row i of the constraint matrix corresponds to rows i and $i+1$ of the corresponding mechanism. Every entry of $C(i, r)$, for $i \in N \setminus n$, $r \in N$ takes on exactly one of four values. If $x_{ir} = x_{(i+1)r} = 0$ then $C(i, r) = Z$. If $x_{ir}, x_{(i+1)r} \neq 0$, then there are three possibilities. If $\alpha \cdot x_{ir} = x_{(i+1)r}$ then $C(i, r) = \downarrow$. If $x_{ir} = \alpha \cdot x_{(i+1)r}$ then $C(i, r) = \uparrow$. Otherwise $C(i, r) = S$.

Example 5.2 Figure 2 shows an optimal, $1/2$ -differentially private mechanism for a specific user. Figure 3 lists the constraint matrix of this mechanism. This mechanism can be derived from the $1/2$ -geometric mechanism by mapping every negative number to 0, every number larger than 5 to 5, 1 to 2 and the other numbers to themselves.

The constraint matrix is well defined: since $\alpha < 1$, at most one of $\alpha \cdot x_{ij} = x_{(i+1)j}$ or $x_{ij} = \alpha \cdot x_{(i+1)j}$ holds (or else both are zero). Also, using that $\alpha > 0$, $x_{ij} = 0$ implies that $x_{kj} = 0$ for all k . Thus every column of C is either all Z 's, which we then call a *Z-column*, or has no Z entries, which we then call a *non-Z column*.

By definition, the constraint matrix encodes the state of all the privacy constraints (3) and (4). The constraint matrix also implicitly encodes the state of the inequality constraints (6). First, $x_{i,j} > 0$ if and only if the j th column of C is non- Z . Also, we can assume that $x_{ij} < 1$ for all $i, j \in N$; otherwise, since $\alpha > 0$, x has singleton support and the proof of the theorem is trivial. Since vertices are uniquely identified by the set of all the constraints that are tight at the vertex, we have the following.

Observation 5.3 *Every mechanism that is a vertex of the polytope of the user-specific LP has a unique constraint matrix.*

Let s denote the number of S entries of C . Let s_i denote the number of S entries in the i th non- Z column. Let $S_i = \sum_{k \leq i} s_i$. Let z denote the number of Z -columns of C . The next few

lemmas, Lemma 5.4, 5.5 and 5.6, use the fact that x is an optimal mechanism to show that the rows of C must exhibit certain structure. Corollary 5.7 and Lemma 5.8 additionally use that x is a vertex to show that the total number of S entries of C is equal to the number of Z -columns. Unless otherwise stated, we ignore Z -columns of C . The following lemma holds for every feasible mechanism.

Lemma 5.4 *No row of the constraint matrix is either all \downarrow 's or all \uparrow 's.*

Proof: Suppose row i of the constraint matrix is all \downarrow 's. So, for all $r \in N$, $\alpha \cdot x_{ir} = x_{(i+1)r}$ and because $\alpha < 1$, $x_{ir} > x_{(i+1)r}$. Since $\sum_r x_{ir} = 1$, $\sum_r x_{(i+1)r} < 1$, contradicting the feasibility of the mechanism. The other case is analogous. \square

The next, key lemma relies on the monotonicity of the loss function l and the optimality of x .

Lemma 5.5 *Every row of the constraint matrix C has the following pattern: some \downarrow 's followed by at most one S followed by some \uparrow 's.*

Proof: Suppose not; then there exists a row index i and column indices $l < m$ such that $C(i, l) \neq \downarrow$ and $C(i, m) \neq \uparrow$. There are two cases: $i \leq (l + m)/2$ and $i > (l + m)/2$. We prove the first case, and suggest modifications to the proof for the second. We now define a feasible mechanism x' with strictly smaller expected loss, contradicting the optimality of x . We multiply the numbers $x_{i'm}$ for all $i' \leq i$ by a $1 - \delta$ factor for some small $\delta > 0$ and set $x_{i'l} = x_{i'l} + \delta \cdot x_{i'm}$.

Because $i \leq (l + m)/2$, $|i' - l| \leq |i' - m|$, and, so, for all $i' \leq i$ the expected loss strictly decreases for strictly monotone loss functions and priors with support N .³ We now discuss feasibility x' . Notice that we modify only two columns of x . The set of constraints (5) are preserved by feasibility of x and the definition of x' . Because $\alpha > 0$, $x_{i'm} > 0$ for all $i' \in N$ and for sufficiently small δ the set of constraints (6) continue to hold. For all $j < i$, privacy constraints that involve the numbers $\{x_{jm}\}$ continue to hold as all we are doing is scaling. For all $j < i$, privacy constraints that only involve the numbers in the sets $\{x_{jm}\}$ and $\{x_{jl}\}$ continue to hold since differential privacy is preserved by scaling and adding. Finally, the privacy constraints involving the numbers $x_{im}, x_{(i+1)m}, x_{il}, x_{(i+1)l}$ are preserved for sufficiently small δ since $C(i, l) \neq \downarrow$ and $C(i, m) \neq \uparrow$.

For the second case, when $i > (l + m)/2$, we interchange the roles of the columns l and m and modify rows $i' \geq i + 1$ rather than $i' \leq i$, i.e., we multiply $x_{i'l}$ for all $i' \geq i + 1$ by $1 - \delta$ for a small, positive δ and set $x_{i'm} = x_{i'm} + \delta \cdot x_{i'l}$. The proof can be modified accordingly. \square

The next lemma relates adjacent rows of C ; it uses the previous lemma combined with the fact that each row of x is a valid probability distribution.

Lemma 5.6 *For all $i \in 0 \dots n - 2$, row $i + 1$ of C has at least one more \downarrow than row i of C ; unless row $i + 1$ has a S in which case row $i + 1$ has at least as many \downarrow 's as row i .*

Proof: Suppose to the contrary that the pair of rows $i, i + 1$ violates the condition in the lemma statement, then we will show that row $i + 2$ of the mechanism violates the probability constraint (5), i.e. $\sum_k x_{(i+2)k} > 1$.

Recall the pattern of a row i of C from Lemma 5.5. Suppose that row i has \downarrow 's in exactly positions $0 \dots j - 1$. Let $\sum_{k < j} x_{ik} = a$, $x_{ij} = b$, $\sum_{k > j} x_{ik} = c$ and $x_{(i+1)j} = b'$. Because row i of x

³If the prior does not have full support or if the loss function is not strictly monotone, we can reach the same conclusion using perturbations and a limiting argument.

must satisfy the probability constraint (5), we have

$$a + b + c = 1. \quad (7)$$

By assumption, $C(i, k) = \downarrow$, for all $k < j$. By Lemma 5.5, we have that for all $k \in j + 1 \dots n$, $C(i, k) = \uparrow$ and $C(i, j)$ is either a S or a \uparrow . So, $\sum_{k < j} x_{(i+1)k} = \alpha \cdot a$, and $\sum_{k > j} x_{(i+1)k} = c/\alpha$. Because row $i + 1$ of x must satisfy the probability constraint (5), we have

$$\alpha \cdot a + b' + c/\alpha = 1. \quad (8)$$

By assumption the rows $i, i + 1$ violate the lemma condition. So, by Lemma 5.5 and the definition of the index j , we have $C(i + 1, k) = \uparrow$ for all columns $k \geq j$. So, $\sum_{k \geq j} x_{(i+2)k} = 1/\alpha \sum_{k \geq j} x_{(i+1)k} = b'/\alpha + c/\alpha^2$. Suppose that for each $k \in 1 \dots j - 1$, $x_{(i+2)k}$ is as small as possible subject to the privacy constraint (so $x_{(i+2)k} = \alpha \cdot x_{(i+1)k}$); this is the worst case for our argument. Then, $\sum_{k < j} x_{(i+2)k} = \alpha \sum_{k < j} x_{(i+1)k} = \alpha^2 \cdot a$. To complete the proof we show that $\alpha^2 a + b'/\alpha + c/\alpha^2$ is strictly larger than 1 and so row $i + 2$ of x violates the probability constraint.

Using Equations (7) and (8) to eliminate a and c , we can show that $\alpha^2 a + b'/\alpha + c/\alpha^2 = 1/\alpha + \alpha - 1 + b - b'\alpha$, which is at least $1/\alpha + \alpha - 1$ because $b - b'\alpha \geq 0$ (x is α -differentially private). Simple algebra shows that $1/\alpha + \alpha - 1 > 1$ whenever $(\alpha - 1)^2 > 0$, which holds since $\alpha < 1$. \square

The next two lemmas relate the number of Z -columns of C to the number of S entries in it.

Corollary 5.7 *The number s of S entries of C is at least the number z of Z -columns of C .*

Proof: Suppose there are a \downarrow 's and b S 's in the first row of the constraint matrix. From Lemma 5.4, the first row cannot consist entirely of \uparrow 's, i.e. $a + b \geq 1$. By Lemma 5.6, the number of \downarrow 's in the last row is at least $a + (n - 1) - (s - b)$. i.e. at least $n - s$. By Lemma 5.4, the number of \downarrow 's in the last row must be at most $n - z$. Chaining the two inequalities gives us the result. \square

Unlike the previous lemmas, the next lemma uses the fact that x is a vertex.

Lemma 5.8 *The number s of S entries of C is equal to the number z of Z -columns of C .*

Proof: By Corollary 5.7 all we need to show is that $s \leq z$. Recall that x is the solution to an LP with $(n + 1)^2$ -variables. Since x is a vertex of the polytope of the user-specific LP, it must be at the intersection of at least $(n + 1)^2$ linearly independent constraints.

Let us account for them. The $n + 1$ constraints (5) are tight. Exactly $z(n + 1)$ constraints of the type specified by (6) are tight—these involve variables x_{ij} , for all $i \in N$, such that j is an index of some Z -column of C . Thus, of the remaining privacy constraints, at least $(n + 1)(n - z)$ must be tight. Because (by definition of C) every such tight constraint corresponds to a unique \downarrow or \uparrow entry, there are at least $(n + 1)(n - z)$ such entries. Thus, of the $n(n + 1 - z)$ entries which are not Z 's, at most $n(n + 1 - z) - (n + 1)(n - z) = z$ of the entries are S . \square

Lemma 5.9 leverages the conditions on the rows of C established by previous lemmas to establish conditions on the columns of C via a counting argument. This completes the second step of the proof of Theorem 3.1.

Lemma 5.9 *For all $i \in 0 \dots n - z$, the i th non- Z column of C has \uparrow 's in rows indexed $0 \dots i - 1 + S_{i-1}$, S 's in the s_i positions that follow and \downarrow 's in the remaining positions.*

Proof: We do induction on non- Z columns. The base case: The first column does not contain any \uparrow 's, because then by Lemma 5.5, an entire row contains only \uparrow 's, contradicting Lemma 5.4. By Lemma 5.6 and Lemma 5.5, a S cannot follow a \downarrow and a \uparrow cannot follow a S in the same column. Therefore, the first column contains s_0 S 's, followed by \downarrow 's.

For the inductive step, assume column $i-1$ has \uparrow 's and S 's in rows $0 \dots i-2+S_{i-1}$ and so column i has \uparrow 's in these positions by Lemma 5.5. We now show by contradiction that $C(i-1+S_{i-1}, i) = \uparrow$; assume otherwise. Let the indicator variable I be 1 if $C(i-1+S_{i-1}, i) = \downarrow$ and 0 if $C(i-1+S_{i-1}, i) = S$. By the induction hypothesis, $C(i-1+S_{i-1}, i-1) = \downarrow$ and so by Lemma 5.5, row $i-1+S_{i-1}$ has at least $i+I$ \downarrow 's (we index from 0). By Lemma 5.5, every one of the rows from row $i+S_{i-1}$ to row $n-1$ must either contain an additional \downarrow or expend one of the $s-S_{i-1}-1+I$ remaining S 's. Thus row n has at least $i+I+n-i-s+1-I = n+1-s$ \downarrow 's. By Corollary 5.8, $s = z$ and so the last row of C is all \downarrow 's, which contradicts Lemma 5.4.

We next show that $C(i+S_{i-1}, i) \neq \uparrow$ and so with an argument similar to the base case, rows $i+S_{i-1} \dots n$ of column i consists of s_i zeros, followed by \downarrow 's, concluding the proof of the induction step. Suppose $C(i+S_{i-1}, i) = \uparrow$. By the induction hypothesis, $C(i-1+S_{i-1}, i-1) = \downarrow$, thus row $i-1+S_{i-1}$ contains at least i \downarrow 's. By induction $C(i+S_{i-1}, i-1) = \downarrow$. So by Lemma 5.5, row $i+S_{i-1}$ does not contain a S and contains i \downarrow 's; this contradicts Lemma 5.6. \square

This concludes the second step of the proof. For the third step of the proof, we work with a finite-range version of the geometric mechanism x^G , called G . G is $w \circ x^G$, where w is a deterministic remap that maps all negative integers to zero, all integers larger than n to n , and all other integers to themselves. We show, constructively, that there exists a remap \hat{y}^u such that $x = \hat{y}^u \circ x^G$. This proves the theorem because the composed remap $\hat{y}^u \circ w$ applied to x^G induces x : $x = (\hat{y}^u \circ w) \circ x^G$.⁴ Lemma 5.10 clarifies the structure of columns of mechanisms induced by deterministic remaps of G . Lemma 5.11 combines Lemma 5.10 and Lemma 5.9 to prove Theorem 3.1.

Lemma 5.10 *Consider a deterministic map that maps the integer sequence $a \dots b$ (and no others) to a fixed $l \in N$. Then, the l th column of the constraint matrix of the mechanism induced by applying this map to G has \uparrow 's in rows $0, \dots, a-1$, S 's in rows $a, \dots, b-1$, and \downarrow 's in the remaining rows.*

Proof: Let the variables $\{g_{ij}\}$ denote the entries of G and let w_i be the entry of in row i of column l of the induced mechanism. By definition, $w_i = \sum_{j=a}^b g_{ij}$.

The j th column of G is a multiple of the column vector $(\alpha^j, \alpha^{j-1}, \dots, \alpha^0, \dots, \alpha^{n-j-1}, \alpha^{n-j})$. So, for i, j , $0 \leq i \leq a-1$, $a \leq j \leq b$, $g_{ij} = \alpha g_{(i+1)j}$, and so, $w_i = \alpha w_{i+1}$. Similarly, for i, j , $b \leq i \leq n$, $a \leq j \leq b$ $g_{ij} = (1/\alpha)g_{i+1j}$, and so, $w_i = (1/\alpha)w_{i+1}$.

Finally, for every i , $a \leq i \leq b-1$, there exist j, j' , $a \leq j, j' \leq b$, such that $g_{ij} = \alpha g_{i+1j}$ and $g_{ij'} = (1/\alpha)g_{i+1j'}$. Also $\alpha < 1$. Therefore, for i such that $a \leq i \leq b-1$ we have that $\alpha w_i < w_{i+1} < (1/\alpha)w_i$. We have the proof by the definition of constraint matrices. \square

The following lemma shows that there exists a remapping of G that induces a mechanism with constraint matrix identical to C , the constraint matrix corresponding to our fixed (vertex) optimal mechanism x . By Observation 5.3, the induced mechanism must be x and the theorem is proved.

Lemma 5.11 *There exists a remapping of G such that the induced mechanism has a constraint matrix identical to C .*

⁴Incidentally, there do exist vertices of the polytope of the user-specific LP that are not derivable from G by remapping. Fortunately, these vertices are not optimal for any user.

Proof: Define the following (user-specific) deterministic map \hat{y} , which we apply to G . Let k_i be the index of the i th non- Z column of C . Map the integers in $i + S_{i-1} \dots i + S_i$ to k_i . We check that \hat{y} is well defined: $S_i + 1$ responses of G are mapped to k_i , and so, a total of $n + 1 - z + s$ distinct responses are mapped. By Lemma 5.8, $z = s$, so the map is well defined for every member of the range N of G .

No integer that is an index of a Z -column of C is in the range of the map. So, the constraint matrix of the induced mechanism has the same set of Z -columns as C . By Lemma 5.9, the i th non- Z column has \uparrow 's in rows $0, \dots, i - 1 + S_{i-1}$, S 's in rows $i + S_{i-1}, \dots, i - 1 + S_i$, and \downarrow 's in the remaining rows. By Lemma 5.10 and the definition of \hat{y} , this is precisely the pattern of the i th non- Z column of the constraint matrix of the induced mechanism. \square

6 Discussion

6.1 Uniqueness

Are there other mechanisms for which an analog of Theorem 3.1 holds? An obvious candidate is the well-studied Laplace mechanism (release the result with noise added from a distribution with density function $\epsilon/2 \cdot e^{-\epsilon|t|}$, where $\alpha = e^{-\epsilon}$), essentially a continuous version of the geometric mechanism. Here is an example where the Laplace mechanism compares unfavorably with the geometric mechanism. Fix a count query, a database size n , and a user with loss function l_{err} and prior $(1/2, 1/2)$ over the two possible results $\{0, 1\}$. The geometric mechanism has (optimal) expected loss $\alpha/(1+\alpha)$ whereas the Laplace mechanism has expected loss $\sqrt{\alpha}/2$. The approximation achieved by the Laplace mechanism tends to ∞ as α approaches 0. Though somewhat pathological, this example rules out the Laplace mechanism as an answer to the above question.

So, is the α -geometric mechanism the unique mechanism for which an analog of Theorem 3.1 holds? No, because, the proofs from the previous section demonstrate that the range-restricted variant G also satisfies the guarantee of the theorem. But a uniqueness result is possible: if we restrict attention to mechanisms with range N , then G is the unique simultaneously optimal mechanism, up to a permutation of the range.

Theorem 6.1 *Fix a database size $n \geq 1$ and a privacy level $\alpha \in [0, 1]$. Suppose there is an α -differentially private mechanism x with range N such that there exists remap y^u for every user u where $y^u \circ x$ minimizes the expected loss of user u (Eq 1) over all oblivious, α -differentially private mechanisms with range N . Then, there exists permutation remap p such that $p \circ x = x^G$.*

Proof: Consider a user with uniform prior over N and the loss function l_{bin} . We can solve the user-specific LP to show that G is the unique optimal mechanism for the user, where the user is constrained to accept the mechanism's responses at face value. Suppose there is a mechanism G' distinct from G that is optimal for all users, and in particular for this user. As G in the above sense, G' must induce G on application of the user's optimal remap. By assumption, the range of G' is N , while G is onto N . Recall that optimal remaps are deterministic and so G' must also be onto N and the optimal remap must be a permutation. \square

6.2 Obliviousness

Recall that our main result (Theorem 3.1) compares the utility of the remapped α -geometric mechanism only to oblivious mechanisms. While natural mechanisms (such as the Laplace mechanism from [10]) are usually oblivious, we now justify this restriction from first principles. Suppose we deploy non-oblivious mechanisms. Measuring the expected utility of such a mechanism for a given user requires the user to have a prior over databases, and not merely over query results. If a user begins only with a prior over query results (arguably the most natural assumption if that's all it cares about), the prior could be extended to one over databases in numerous ways. Singling out any one extension would be arbitrary, so we consider optimizing worst-case utility over all such extensions. Formally, for Lemma 6.2 below, we temporarily replace (1) with the objective function

$$\max_{p_d \rightarrow p_i} \sum_{d \in D^n} p_d \sum_{r \in N} x_{dr} \cdot l(f(d), r). \quad (9)$$

In (9), $p_d \rightarrow p_i$ indicates that p_d is a prior over D^n that induces the user's prior p_i over N , meaning $p_i = \sum_{d|f(d)=i} p_d$ for every $i \in N$. The following lemma then shows that the restriction to oblivious mechanisms is without loss of generality.

Proposition 6.2 *Fix a database size $n \geq 1$ and privacy level α . For every user with prior $\{p_i\}$ and loss function l , there is an α -differentially private mechanism that minimizes the objective function (9) and is also oblivious.*

Proof: Fix a privacy level α , a database size n and a user with prior $\{p_i\}$ and monotone loss function l . Let x be a mechanism that minimizes (9) for this user. We define a mechanism x' that is oblivious, α -differentially private, and has at most as much expected loss as x for the objective (9). For any database $d \in D^n$, define $E(d)$ as the set of databases with the same query result as d . For a database $d \in D^n$ and response $r \in N$, let x'_{dr} be the average of $x_{d'r}$ over the databases $\{d' | d' \in E(d)\}$; x' is oblivious by definition and also a valid mechanism, in that it specifies a distribution over responses for every underlying database.

We now show that x' is α -differentially private. Fix two databases $d_1, d_2 \in D^n$ such that d_1 and d_2 differ in exactly one row; We need to show that $\alpha x'_{d_1 r} \leq x'_{d_2 r}$. Assume $f(d_1) \neq f(d_2)$, otherwise the proof is trivial.

For any database $d \in E(d_1)$, we can generate all its neighbors (databases that differ in exactly one row) in $E(d_2)$ by enumerating all the ways in which we can change the query result by exactly 1. For instance when $f(d_1) = f(d_2) + 1$, pick one of the $n - f(d_1)$ rows that satisfy the predicate in d_1 and change its value to one of those that violates the predicate. This process is identical for all databases of $E(d_1)$, and so for all $d \in E(d_1)$, the number of neighbors of d that belong to the set $E(d_2)$ is the same (does not vary with d). Similarly, for all $d \in E(d_2)$, the number of neighbors of d that belong to the set $E(d_1)$ is the same.

Consider the following set of inequalities that hold because x is α -differentially private: $d \in E(d_1)$, $d' \in E(d_2)$, where d_1 and d_2 are neighbors, $\alpha x_{dr} \leq x_{d'r}$. By the argument in the above paragraph, all the databases in $E(d_1)$ appears equally frequently in the left-hand-side of the above inequality and all the databases in $E(d_2)$ equally frequently in the right-hand-side. Summing the inequalities and recalling the definition of x' completes the proof of privacy.

We now show that x' has better worst case expected loss. Since x' is oblivious, the expected loss is the same for all prior distributions over databases that induces the prior $\{p_i\}$ over results.

Set	1	2
$\{\}$		
$\{1\}$	11/12	1/12
$\{2\}$	2/3	1/3
$\{3\}$		
$\{1, 2\}$		
$\{1, 3\}$	5/6	1/6
$\{2, 3\}$	1/3	2/3
$\{1, 2, 3\}$		

Figure 4: The mechanism x_1

By definition of x' , the expected loss is: $\sum_{i \in N} p_i \text{avg}_{d|f(d)=i} \sum_{r \in N} x_{dr} \cdot l(i, r)$. For the mechanism x , we can construct an adversarial distribution over databases that induces $\{p_i\}$, such that for every $i \in N$, the weight p_i is assigned entirely to a database d that maximizes $\sum_r x_{dr} \cdot l(f(d), r)$ over $d|f(d) = i$. Thus x incurs at least as much expected loss as x' . \square

An alternative is to model users as having a prior over databases. Though priors on databases induce priors on query results, the converse is not necessarily true. Formally, for Theorem 6.3 below, we use the following objective function.

$$\sum_{d \in D^n} p_d \sum_{r \in N} x_{dr} \cdot l(f(d), r). \quad (10)$$

We show that no analog of Theorem 3.1 is possible in this model.⁵

Theorem 6.3 *There exists a database size, level of privacy and two users, each with monotone loss functions and distinct priors over databases such that no mechanism is simultaneously optimal for both of them.*

We now prove Theorem 6.3. Suppose that the domain D is $\{0, 1\}$ and we draw databases from D^3 . Fix a count query that counts the number of rows that are 1. Fix a privacy level $\alpha = 1/2$. We label databases by the set of rows that have values 1; in this notation, the result of the query is the cardinality of the set. The first user's prior over databases is $\frac{1}{4}$ on $\{1\}$ and $\{2\}$, $\frac{1}{2} - \epsilon$ on $\{2, 3\}$, ϵ on $\{1, 3\}$ and 0 on all others. The second user's prior is defined by interchanging the roles of rows 1 and 2 in this definition. Both users have the monotone loss function $l(i, r) = |i - r|^{1-\delta}$. There exist small, positive ϵ and δ , for which the unique (non-oblivious) optimal mechanism (where the user accepts results at face value) for the first user can be shown to be the mechanism x_1 defined in Figure 6.2. (Only the rows that referred to by our proof are specified.) Similarly, the unique optimal mechanism for the second user can be derived by interchanging the roles of the rows 1 and 2.

We will show that there is no $1/2$ -differentially private mechanism x that implements x_1 and x_2 simultaneously. Assume to the contrary: Let y_1 and y_2 be deterministic maps that when applied

⁵Proposition 6.2 implicitly shows that with the restriction to oblivious mechanisms, Theorem 3.1 would continue to hold in this model. However, Theorem 6.3 implies that obliviousness is no longer without loss of generality.

Set	l	m	n	o
$\{\}$				
$\{1\}$	$\frac{1}{12}(8 - \delta_1)$	$\frac{1}{12}(1 - \delta_1)$	$\frac{1}{12}(3 + \delta_1)$	$\frac{1}{12}\delta_1$
$\{2\}$	$\frac{1}{12}(8 - \delta_2)$	$\frac{1}{12}(1 - \delta_2)$	$\frac{1}{12}\delta_2$	$\frac{1}{12}(3 + \delta_2)$
$\{3\}$				
$\{1, 2\}$				
$\{1, 3\}$	$\frac{1}{6}(2 - \delta_3)$	$\frac{1}{6}(1 - \delta_3)$	$\frac{1}{6}(3 + \delta_3)$	$\frac{1}{6}\delta_3$
$\{2, 3\}$	$\frac{1}{6}(2 - \delta_4)$	$\frac{1}{6}(1 - \delta_4)$	$\frac{1}{6}\delta_4$	$\frac{1}{6}(3 + \delta_4)$
$\{1, 2, 3\}$				

Figure 5: The mechanism M

to x induce x_1 and x_2 respectively. (We skip the easy modification to the proof that allows y_1 and y_2 to be randomized.) We first show that x , y_1 and y_2 have the following form:

Lemma 6.4 *Without any loss of generality, x has a range $R = \{l, m, n, o\}$ of size 4, y_1 maps l and n to 1 and m and o to 2 and y_2 maps l and o to 1 and m and n to 2.*

Proof: Both x_1 and x_2 have range size 2. Label each member of the range of x in one of four ways based on where it is mapped by y_1 (1 or 2) and y_2 (1 or 2). All the range points with the same label may be combined into one range point. For instance o consists of all the range points mapped to 2 by y_1 and 1 by y_2 . The proof is complete. \square

We are now ready to prove Theorem 6.3.

Proof of Theorem 6.3: By the previous lemma and the definitions of x_1 and x_2 , the mechanism x must have the form in Figure 6.2. Because x is $1/2$ -differentially private and as $\{1\}$ and $\{2\}$ differ in exactly two rows, columns n and o yield the following inequalities: $3 + \delta_1 \leq 4\delta_2$, and $3 + \delta_2 \leq 4\delta_1$. Adding the two inequalities, we have: $6 \leq 3(\delta_1 + \delta_2)$. Because all the entries of x are probabilities, we have $0 \leq \delta_1, \delta_2 \leq 1$. So it must be that $\delta_1 = \delta_2 = 1$. By a similar argument applied to the databases $\{3\}$ and $\{4\}$, we can show that $\delta_3 = \delta_4 = 1$.

Thus, the probability masses on l when the underlying databases are $\{1\}$ and $\{1, 3\}$ are $7/12$ and $1/6$ respectively. But this violates privacy because, the two databases differ in exactly one row but the two probabilities are not within a factor 2 of each other. \square

7 Future Directions

We proposed a model of user utility, where users are parametrized by a prior (modeling side information) and a loss function (modeling preferences). Theorem 3.1 shows that for every fixed count query, database size, and level of privacy, there is a single simple mechanism that is simultaneously optimal for all rational users. Are analogous results possible for other definitions of privacy, such as the additive variant of differential privacy (see [9])? Is an analogous result possible for other types of queries or for multiple queries at once? When users have priors over databases (Theorem 6.3), are any positive results (such as simultaneous *approximation*) achievable via a single mechanism?

8 Acknowledgments

We thank Preston McAfee, John C. Mitchell, Rajeev Motwani, David Pennock and the anonymous referees.

References

- [1] L. Backstrom, C. Dwork, and J. Kleinberg. Wherefore art thou r3579x?: anonymized social networks, hidden patterns, and structural steganography. In *Proceedings of the 16th International Conference on World Wide Web (WWW)*, pages 181–190, 2007.
- [2] B. Barak, K. Chaudhuri, C. Dwork, S. Kale, F. McSherry, and K. Talwar. Privacy, accuracy, and consistency too: a holistic solution to contingency table release. In *Proceedings of the 26th ACM SIGACT-SIGMOD-SIGART Symposium on Principles of Database Systems (PODS)*, pages 273–282, 2007.
- [3] D. Bertsimas and J. N. Tsitsiklis. *Introduction to Linear Optimization*. Athena Scientific, 1997.
- [4] A. Blum, C. Dwork, F. McSherry, and K. Nissim. Practical privacy: The SuLQ framework. In *Proceedings of the 24th ACM SIGACT-SIGMOD-SIGART Symposium on Principles of Database Systems (PODS)*, pages 128–138, 2005.
- [5] A. Blum, K. Ligett, and A. Roth. A learning theory approach to non-interactive database privacy. In *Proceedings of the 40th Annual ACM Symposium on Theory of Computing (STOC)*, pages 609–618, 2008.
- [6] U. S. Census Bureau. The 2008 statistical abstract. <http://www.census.gov/compendia/statab/>.
- [7] I. Dinur and Nissim K. Revealing information while preserving privacy. In *Proceedings of the 22nd ACM SIGACT-SIGMOD-SIGART Symposium on Principles of Database Systems (PODS)*, pages 202–210, 2003.
- [8] C. Dwork. Differential privacy. In *Proceedings of the 33rd Annual International Colloquium on Automata, Languages, and Programming (ICALP)*, volume 4051 of *Lecture Notes in Computer Science*, pages 1–12, 2006.
- [9] C. Dwork. Differential privacy: A survey of results. In *5th International Conference on Theory and Applications of Models of Computation (TAMC)*, volume 4978 of *Lecture Notes in Computer Science*, pages 1–19, 2008.
- [10] C. Dwork, F. McSherry, K. Nissim, and A. Smith. Calibrating noise to sensitivity in private data analysis. In *Third Theory of Cryptography Conference (TCC)*, volume 3876 of *Lecture Notes in Computer Science*, pages 265–284, 2006.
- [11] C. Dwork, F. McSherry, and K. Talwar. The price of privacy and the limits of LP decoding. In *Proceedings of the 39th Annual ACM Symposium on Theory of Computing (STOC)*, pages 85–94, 2007.

- [12] C. Dwork and K. Nissim. Privacy-preserving datamining on vertically partitioned databases. In *24th Annual International Cryptology Conference (CRYPTO)*, volume 3152 of *Lecture Notes in Computer Science*, pages 528–544, 2004.
- [13] S. P. Kasiviswanathan, H. K. Lee, K. Nissim, S. Raskhodnikova, and A. Smith. What can we learn privately? In *Proceedings of the 49th Annual IEEE Symposium on Foundations of Computer Science (FOCS)*, pages 531–540, 2008.
- [14] S. P. Kasiviswanathan and A. Smith. A note on differential privacy: Defining resistance to arbitrary side information. <http://arxiv.org/abs/0803.3946v1>, 2008.
- [15] A. Mas-Colell, M. D. Whinston, and J. R. Green. *Microeconomic Theory*. Oxford University Press, New York, 1995.
- [16] F. McSherry and K. Talwar. Mechanism design via differential privacy. In *Proceedings of the 48th Annual IEEE Symposium on Foundations of Computer Science (FOCS)*, pages 94–103, 2007.
- [17] A. Narayanan and V. Shmatikov. Robust de-anonymization of large sparse datasets. In *Proceedings of the 2008 IEEE Symposium on Security and Privacy (SP)*, pages 111–125, 2008.
- [18] K. Nissim, S. Raskhodnikova, and A. Smith. Smooth sensitivity and sampling in private data analysis. In *Proceedings of the 39th Annual ACM Symposium on Theory of Computing (STOC)*, pages 75–84, 2007.
- [19] Wikipedia. AOL search data scandal. http://en.wikipedia.org/wiki/AOL_search_data_scandal.